

## Half-quantum linear algebra

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The Cayley–Hamilton–Newton theorem for half-quantum matrices is proven.

*Keywords:* Yang–Baxter, quantum matrix, Cayley–Hamilton, Newton identities

### 1. Introduction

The Cayley–Hamilton theorem in the linear algebra and the Newton identities in the theory of symmetric functions have a common predecessor, discovered in<sup>3</sup> and called there Cayley–Hamilton–Newton (CHN) identities. These are matrix identities, which naturally include the Cayley–Hamilton identity and whose trace reproduces the Newton identities. The CHN identities admit a  $q$ -deformation, see Ref. 3. Later,<sup>4</sup> the CHN identities were established for a larger class of quantum matrix algebras which includes both the algebra of functions on a quantum group of  $GL$  type and the deformed universal enveloping algebra of  $\mathfrak{gl}$  type. This class of algebras, defined with the help of a compatible pair  $(\hat{R}, \hat{F})$  of  $R$ -matrices (see the precise definition in Section 2), can be described differently, as braided Hopf algebras universally coacting (in the braided sense) on two, left and right, quantum spaces. When the  $R$ -matrix  $\hat{F}$  is the flip, this is the usual universally coacting Hopf algebra.

There is, nowadays, a certain interest in the algebras universally coacting on just one quantum space, see<sup>1</sup> and the literature cited there. We give the definition of the braided Hopf algebra  $\mathcal{M}_{SA}(\hat{R}, \hat{F})$  universally coacting (again in the braided sense) on one quantum space. The definition appeals again to a compatible pair  $(\hat{R}, \hat{F})$  of  $R$ -matrices. The matrix whose entries generate  $\mathcal{M}_{SA}(\hat{R}, \hat{F})$  is called *half-quantum* (HQ-)matrix. An example of HQ-matrices arises naturally in the theory of quantum groups of  $GL$  type with spectral parameters, see Section 2.

The main result of the present article is the CHN theorem for the HQ-matrices defined by a pair  $(\hat{R}, \hat{F})$  where  $\hat{R}$  is of Hecke type, that is, the spectral decomposition of  $\hat{R}$  has two projectors,  $S$  and  $A$ . The algebras  $\mathcal{M}_{SA}(\hat{R}, \hat{F})$  coact on the “right” quantum space associated to  $S$ . The formal exchange of projectors leads to the statements about the algebras coacting on the “left”

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quantum space, associated to  $S$ , so it is enough to investigate the algebras  $\mathcal{M}_{SA}(\hat{R}, \hat{F})$ .

The coefficients of the CHN identities are analogues of the sets of elementary and complete symmetric functions. Contrary to the algebra generated by the full quantum matrix, none of these sets is commutative for HQ-matrices. As for full quantum matrices, the CHN theorem for HQ-matrices uses modified matrix powers. We define four, in general, different powers,  $M^{\rightarrow k}$  and  $M^{\leftarrow k}$ , of an HQ-matrix  $M$ ; if  $M$  is a full quantum matrix then  $M^{\rightarrow k} = M^{\leftarrow k}$ . As a direct consequence of the CHN theorem we obtain the Cayley–Hamilton theorem and Newton identities for HQ-matrices.

The HQ-matrices can be defined in a more general setting, when one knows which eigenprojectors of an R-matrix belong to the symmetric and anti-symmetric part (see Section 3 in<sup>5</sup> for a general discussion). This is the case, in particular, of the R-matrices for simple Lie groups (in any representation). The CHN theorem for full quantum matrices is known<sup>6</sup> in the simplest situation, for the orthogonal and symplectic groups in the defining representation and it will be interesting to generalize it for the HQ-matrices.

**Notation.** Let  $V$  be a finite-dimensional  $\mathbb{C}$ -vector space and  $V^{\otimes M} := V \otimes V \otimes \dots$  ( $M$  times,  $M = 0, 1, 2, \dots$ ; by convention,  $V^{\otimes 0} := \mathbb{C}$ ). Let  $\mathfrak{A}$  be an associative algebra and  $X$  an  $\mathfrak{A}$ -valued operator on  $V$  (that is, the matrix elements of  $X$  belong to  $\mathfrak{A}$ ). We denote by  $X_j$  the operator which acts as  $X$  on the  $j$ -th copy of the space  $V$  in  $V \otimes V \otimes \dots$  (and as the identity on other copies); for an  $\mathfrak{A}$ -valued operator  $Y$  on  $V \otimes V$ , we denote by  $Y_{j,k}$  the operator which acts as  $Y$  on the  $j$ -th and  $k$ -th copies of the space  $V$  in  $V \otimes V \otimes \dots$  (and as the identity on other copies) *etc.* We write sometimes  $Y_j := Y_{j,j+1}$  for brevity. The identity operator on  $V^{\otimes M}$  we denote by  $\text{Id}$  when the value of  $M$  is clear from the context. We denote by  $P \in \text{Aut}(V \otimes V)$  the permutation operator,  $P(v \otimes w) := w \otimes v$ . The operation of taking traces in copies of  $V$  with the numbers  $i_1 \dots i_k$  is denoted by  $\text{tr}_{(i_1 \dots i_k)}$ . For an  $\mathfrak{A}$ -valued operator  $Z$  on  $V^{\otimes m}$  define the operator  $Z^{\uparrow k}$ ,  $k = 0, 1, 2, \dots$ , in  $V^{\otimes m+k}$  by  $Z^{\uparrow k} := \text{Id}^{\otimes k} \otimes Z$ . We use symmetric  $q$ -numbers,  $j_q := q^{j-1} + q^{j-3} + \dots + q^{-j+1}$  for  $j = 0, 1, \dots$  (by convention, the empty sum is 0). Given a numerical  $q$ , we shall say that a positive integer  $j$  is  $q$ -admissible if  $k_q \neq 0$  for  $k = 1, 2, \dots, j$ .

## 2. Generalized half-quantum matrices

**2.1 Operators  $\hat{R}$  and  $\hat{F}$ .** Generalized half-quantum matrix algebras are defined with the help of two operators  $\hat{R}, \hat{F} \in \text{Aut}(V \otimes V)$  which satisfy, depending on a property in question, to a combination of conditions (i)–(iv) below.

(i)  $(\hat{R}, \hat{F})$  is a *compatible pair* of R-matrices:

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}, \quad \hat{F}_{12}\hat{F}_{23}\hat{F}_{12} = \hat{F}_{23}\hat{F}_{12}\hat{F}_{23}, \quad (1)$$

$$\hat{R}_{12}\hat{F}_{23}\hat{F}_{12} = \hat{F}_{23}\hat{F}_{12}\hat{R}_{23}, \quad \hat{F}_{12}\hat{F}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{F}_{12}\hat{F}_{23}. \quad (2)$$

(ii) The R-matrix  $\hat{R}$  is of *Hecke type*: its projector decomposition is  $\hat{R} = qS^{(2)} - q^{-1}A^{(2)}$  where  $q \in \mathbb{C}^*$  is such that 2 is  $q$ -admissible. The projectors  $S^{(2)}$  and  $A^{(2)}$  (called, respectively,  $q$ -symmetrizer and  $q$ -antisymmetrizer) are complementary,  $S^{(2)} + A^{(2)} = \text{Id}$ .

The higher  $q$ -symmetrizers and  $q$ -antisymmetrizers  $S^{(k+1)}, A^{(k+1)} \in \text{End}(V^{\otimes k+1})$ ,  $k > 1$ ,

can be defined, if  $k + 1$  is  $q$ -admissible, in several ways, *e.g.*, inductively, in any of the forms

$$A^{(k)\uparrow 1} \left( q^k \text{Id} - k_q \hat{R}_1 \right) A^{(k)\uparrow 1} =: (k + 1)_q A^{(k+1)} := A^{(k)} \left( q^k \text{Id} - k_q \hat{R}_k \right) A^{(k)} , \quad (3)$$

$$S^{(k)\uparrow 1} \left( q^{-k} \text{Id} + k_q \hat{R}_1 \right) S^{(k)\uparrow 1} =: (k + 1)_q S^{(k+1)} := S^{(k)} \left( q^{-k} \text{Id} + k_q \hat{R}_k \right) S^{(k)} . \quad (4)$$

By convention,  $A^{(1)} := \text{Id}$  and  $S^{(1)} := \text{Id}$ ; then (3) and (4) hold for  $k = 1$  as well.

(iii) The Hecke R-matrix  $\hat{R}$  is *even of height*<sup>2</sup>  $n$ :  $n$  is  $q$ -admissible,  $\text{rank}(A^{(n)}) = 1$  and  $A^{(n)}(q^n \text{Id} - n_q \hat{R}_n)A^{(n)} = 0$ .

(iv) The R-matrix  $\hat{F}$  is invertible and *skew-invertible*: there exists  $\Psi \in \text{Aut}(V \otimes V)$  satisfying  $\text{tr}_2(\Psi_{12} \hat{F}_{23}) = P_{13}$ .

**2.2 F-trace.** The *quantum trace* of a matrix  $X$  with arbitrary entries is  $\text{tr}_{\hat{F}}(X) := \text{tr}(DX)$ , where  $D \in \text{End}(V)$  is defined by  $D_1 := \text{tr}_2(\Psi_{12})$ . We have

$$\text{tr}_{\hat{F}(2)}(\hat{F}_1) = \text{Id}_1 , \quad \hat{F}_1 D_1 D_2 = D_1 D_2 \hat{F}_1 , \quad \text{tr}_{\hat{F}(2)}(\hat{F}_1^\epsilon X_1 \hat{F}_1^{-\epsilon}) = \text{Id}_1 \text{tr}_{\hat{F}}(X) , \quad \epsilon = \pm 1 .$$

**2.3** Let  $M$  be an operator on  $V$  with arbitrary entries. Define inductively

$$M_{\overline{1}} := M_1 , \quad M_{\overline{k+1}} := \hat{F}_{k,k+1} M_{\overline{k}} \hat{F}_{k,k+1}^{-1} . \quad (5)$$

**Definition.** Let  $\mathcal{M}_{SA}(\hat{R}, \hat{F})$  be the unital associative algebra generated by the entries of  $M$  with the defining relations

$$S_{12}^{(2)} M_{\overline{1}} M_{\overline{2}} A_{12}^{(2)} = 0 . \quad (6)$$

We call  $M$  the *half-quantum SA-matrix* (HQ-matrix, for brevity).

To explain the meaning of the definition, let  $\mathcal{X}_S$  be the algebra generated by components  $\{x^a\}$  of a vector from  $V$  with the defining quadratic relations  $(S^{(2)})_{cd}^{ab} x^c x^d = 0$  (the Einstein summation convention is assumed throughout the text). The HQ-matrix coacts,  $x^a \rightarrow M_b^a x^b$ , on the algebra  $\mathcal{X}_S$  and generates the braided Hopf algebra universally coacting on  $\mathcal{X}_S$ . The braiding between the entries of  $M$  and  $x$  is given by  $X_1 M_2 = M_2 X_1$ , where  $X_b^a := c_b x^a$ ,  $c_b$  are independent commuting variables. As for the braided coproduct, let  $M$  and  $M'$  be HQ-matrices which satisfy

$$\hat{F}_{12} M_1 \hat{F}_{12}^{-1} M'_1 = M'_1 \hat{F}_{12} M_1 \hat{F}_{12}^{-1} .$$

Then  $MM'$  is again an HQ-matrix.

**Notation.** For natural  $l$  and  $k$ ,  $l \leq k$ , set  $M_{\overline{l \rightarrow k}} := M_{\overline{l}} \dots M_{\overline{k}}$ ,  $\hat{F}_{l \rightarrow k} := \hat{F}_l \dots \hat{F}_{k-1}$ ,  $\hat{R}_{l \rightarrow k} := \hat{R}_l \dots \hat{R}_{k-1}$  and  $\hat{R}_{k \leftarrow l} := \hat{R}_{k-1} \dots \hat{R}_l$  (by convention, the empty product is 1).

**2.4** Let  $(\hat{R}, \hat{F})$  be a pair satisfying the conditions (i) and (iv), Subsection 2.1. The following Lemma, valid for a matrix  $M$  with arbitrary entries, was proved in Ref. 4.

**Lemma 1. a)** *We have*

$$\hat{F}_i M_{\overline{k}} = M_{\overline{k}} \hat{F}_i \quad \text{and} \quad \hat{R}_i M_{\overline{k}} = M_{\overline{k}} \hat{R}_i \quad \text{for } k \neq i, i + 1 , \quad (7)$$

$$\hat{F}_{i \rightarrow k+1} M_{\overline{i \rightarrow k}} = M_{\overline{i+1 \rightarrow k+1}} \hat{F}_{i \rightarrow k+1} \quad \text{for } i \leq k . \quad (8)$$

**b)** *Let  $\alpha(Y^{(k)}) := \text{tr}_{\hat{F}(1, \dots, k)}(Y^{(k)} M_{\overline{1 \rightarrow k}})$  where  $Y^{(k)}$  is a polynomial in  $\hat{R}_1, \dots, \hat{R}_{k-1}$ . Then*

$$\text{tr}_{\hat{F}(i+1, \dots, i+k)}(Y^{(k)\uparrow i} M_{\overline{i+1 \rightarrow i+k}}) = \text{Id}_{1, \dots, i} \alpha(Y^{(k)}) . \quad (9)$$

**2.5** Assume now that  $M$  is an HQ-matrix. Since  $\hat{R}_1 = qS^{(2)} - q^{-1}A^{(2)}$  and  $S^{(2)} + A^{(2)} = \text{Id}$ , the relations (6) can be rewritten in the four following equivalent (when  $q + q^{-1} \neq 0$ ) forms

$$\begin{aligned} \hat{R}_1 M_{\overline{1}} M_{\overline{2}} A^{(2)} &= -q^{-1} M_{\overline{1}} M_{\overline{2}} A^{(2)}, \quad S^{(2)} M_{\overline{1}} M_{\overline{2}} \hat{R}_1 = q S^{(2)} M_{\overline{1}} M_{\overline{2}}, \\ A^{(2)} M_{\overline{1}} M_{\overline{2}} A^{(2)} &= M_{\overline{1}} M_{\overline{2}} A^{(2)}, \quad S^{(2)} M_{\overline{1}} M_{\overline{2}} S^{(2)} = S^{(2)} M_{\overline{1}} M_{\overline{2}}. \end{aligned} \quad (10)$$

By induction on  $k = 1, 2, \dots$  we obtain, using Lemma 1,

$$S_{j,j+1}^{(2)} M_{\overline{j}} M_{\overline{j+1}} A_{j,j+1}^{(2)} = 0, \quad (11)$$

$$A^{(k)\uparrow i} M_{\overline{i+1 \rightarrow i+k}} A^{(k)\uparrow i} = M_{\overline{i+1 \rightarrow i+k}} A^{(k)\uparrow i}, \quad i = 0, 1, 2, \dots \quad (12)$$

$$S^{(k)\uparrow i} M_{\overline{i+1 \rightarrow i+k}} S^{(k)\uparrow i} = S^{(k)\uparrow i} M_{\overline{i+1 \rightarrow i+k}}, \quad i = 0, 1, 2, \dots \quad (13)$$

In eqs.(12)-(13),  $k$  is assumed to be  $q$ -admissible.

**2.6 Example.** Let  $(\hat{R}, \hat{F})$  be a compatible pair with  $\hat{R}$  of Hecke type. Let  $\hat{R}(u) := \hat{R} + (q - q^{-1})/(u^2 - 1)$  be the Baxterization of  $\hat{R}$  and let  $T(u)$  be the matrix generating the algebra with the defining relations  $\hat{R}_{12}(u/v)T_1(u)T_2(v) = T_1(v)T_2(u)\hat{R}_{12}(u/v)$  (in the simplest cases, this can be an algebra related to quantum or classical (super)-Yangians or affine algebras). Now,  $\hat{R}(q)$  is proportional to the symmetrizer, so setting  $u = qv$ , multiplying from the right by  $A_{12}^{(2)}$  and writing  $T(qv) = q^{v\partial_v}T(v)q^{-v\partial_v}$  we find  $S_{12}^{(2)}q^{v\partial_v}T_1(v)q^{-v\partial_v}T_2(v)A_{12}^{(2)} = 0$ , which implies that  $q^{-v\partial_v}T$  and  $Tq^{-v\partial_v}$  are HQ-matrices,  $S_{12}^{(2)}q^{-v\partial_v}T_1(v)q^{-v\partial_v}T_2(v)A_{12}^{(2)} = 0$  and  $S_{12}^{(2)}T_1(v)q^{-v\partial_v}T_2(v)q^{-v\partial_v}A_{12}^{(2)} = 0$ .

### 3. Cayley–Hamilton–Newton theorem

**3.1 Symmetric functions.** Let  $M$  be an HQ-matrix. Define four sets,  $\{s_k(M)\}$ ,  $\{\bar{s}_k(M)\}$ ,  $\{\sigma_k(M)\}$  and  $\{\tau_k(M)\}$ ,  $k = 0, 1, 2, \dots$ , of elements of the algebra  $\mathcal{M}_{SA}(\hat{R}, \hat{F})$  by  $s_0(M) = \bar{s}_0(M) = \sigma_0(M) = \tau_0(M) = 1$  and, for  $k > 0$ ,

$$s_k(M) := \text{tr}_{\hat{F}(1 \dots k)}(\hat{R}_{1 \rightarrow k} M_{\overline{1 \rightarrow k}}), \quad \bar{s}_k(M) := \text{tr}_{\hat{F}(1 \dots k)}(\hat{R}_{k \leftarrow 1} M_{\overline{1 \rightarrow k}}), \quad (14)$$

$$\sigma_k(M) := q^k \text{tr}_{\hat{F}(1 \dots k)}(A^{(k)} M_{\overline{1 \rightarrow k}}), \quad \tau_k(M) := q^{-k} \text{tr}_{\hat{F}(1 \dots k)}(S^{(k)} M_{\overline{1 \rightarrow k}}). \quad (15)$$

**3.2 Powers of HQ-matrices.** Define the matrices  $M^{\rightarrow k}$ ,  $\tilde{M}^{\rightarrow k}$ ,  $M^{\leftarrow k}$  and  $\tilde{M}^{\leftarrow k}$ ,  $k$ -th powers of  $M$ , by

$$(M^{\rightarrow k})_1 := \text{tr}_{\hat{F}(2, \dots, k)}(M_{\overline{1 \rightarrow k}} \hat{R}_{1 \rightarrow k}), \quad (M^{\leftarrow k})_1 := \text{tr}_{\hat{F}(2, \dots, k)}(\hat{R}_{k \leftarrow 1} M_{\overline{1 \rightarrow k}}), \quad (16)$$

$$(\tilde{M}^{\rightarrow k})_1 := \text{tr}_{\hat{F}(2, \dots, k)}(\hat{R}_{1 \rightarrow k} M_{\overline{1 \rightarrow k}}), \quad (\tilde{M}^{\leftarrow k})_1 := \text{tr}_{\hat{F}(2, \dots, k)}(M_{\overline{1 \rightarrow k}} \hat{R}_{k \leftarrow 1}). \quad (17)$$

Define the  $k$ -th wedge powers  $M^{\wedge k}$ ,  $\tilde{M}^{\wedge k}$  and the  $k$ -th symmetric powers  $M^{s_k}$ ,  $\tilde{M}^{s_k}$  of  $M$  by

$$(M^{\wedge k})_1 := \text{tr}_{\hat{F}(2, \dots, k)}(A^{(k)} M_{\overline{1 \rightarrow k}}), \quad (M^{s_k})_1 := \text{tr}_{\hat{F}(2, \dots, k)}(M_{\overline{1 \rightarrow k}} S^{(k)}), \quad (18)$$

$$(\tilde{M}^{\wedge k})_1 := \text{tr}_{\hat{F}(2, \dots, k)}(M_{\overline{1 \rightarrow k}} A^{(k)}), \quad (\tilde{M}^{s_k})_1 := \text{tr}_{\hat{F}(2, \dots, k)}(S^{(k)} M_{\overline{1 \rightarrow k}}). \quad (19)$$

**3.3 Cayley–Hamilton–Newton theorem for HQ-matrices** (Cf.<sup>3</sup>) Assume that the pair  $(\hat{R}, \hat{F})$  satisfies conditions (i), (ii) and (iv), Subsection 2.1, and assume that the positive integer

$j$  is  $q$ -admissible. The HQ-matrix satisfies the following matrix identities

$$j_q M^{\wedge j} = \sum_{k=0}^{j-1} (-1)^{j-k+1} M^{\leftarrow j-k} \sigma_k(M) , \quad (20)$$

$$j_q M^{sj} = \sum_{k=0}^{j-1} M^{\rightarrow j-k} \tau_k(M) , \quad (21)$$

$$j_q \tilde{M}^{\wedge j} = \sum_{k=0}^{j-1} (-1)^{j-k+1} \tilde{M}^{\leftarrow j-k} \sigma_k(M) , \quad (22)$$

$$j_q \tilde{M}^{sj} = \sum_{k=0}^{j-1} \tilde{M}^{\rightarrow j-k} \tau_k(M) . \quad (23)$$

We sketch the *proof* of (20). For  $k$ ,  $1 \leq k < j$ , rewrite  $(M^{\leftarrow j-k} \sigma_k(M))_1$  as

$$\begin{aligned} & q^k \text{tr}_{\hat{F}(2 \dots j-k)} (\hat{R}_{j-k \leftarrow 1} M_{1 \rightarrow j-k} \overline{M_{j-k+1 \rightarrow j}}) \text{tr}_{\hat{F}(1 \dots k)} (A^{(k)} M_{1 \rightarrow k}) \\ &= q^k \text{tr}_{\hat{F}(2 \dots j)} (A^{(k) \uparrow j-k} \hat{R}_{j-k \leftarrow 1} M_{1 \rightarrow j-k} \overline{M_{j-k+1 \rightarrow j}}) = q^k \text{tr}_{\hat{F}(2 \dots j)} (A^{(k) \uparrow j-k} \hat{R}_{j-k \leftarrow 1} M_{1 \rightarrow j}) . \end{aligned} \quad (24)$$

Here we used (9). We use the recurrence for the  $q$ -antisymmetrizers in the form  $q^k A^{(k) \uparrow 1} = (k+1)_q A^{(k+1)} + k_q A^{(k) \uparrow 1} \hat{R}_1 A^{(k) \uparrow 1}$  to rewrite the last expression in (24) as

$$\begin{aligned} & (k+1)_q \text{tr}_{\hat{F}(2 \dots j)} (A^{(k+1) \uparrow j-k-1} \hat{R}_{j-k \leftarrow 1} M_{1 \rightarrow j}) \\ &+ k_q \text{tr}_{\hat{F}(2 \dots j)} (A^{(k) \uparrow j-k} \hat{R}_{j-k} A^{(k) \uparrow j-k} \hat{R}_{j-k \leftarrow 1} M_{1 \rightarrow j}) . \end{aligned} \quad (25)$$

In the second term of (25), the right  $A^{(k) \uparrow j-k}$  commutes with  $\hat{R}_{j-k \leftarrow 1}$  and the left  $A^{(k) \uparrow j-k}$  can be cyclically moved. So, the last term takes the form

$$\begin{aligned} & k_q \text{tr}_{\hat{F}(2 \dots j)} (\hat{R}_{j-k+1 \leftarrow 1} A^{(k) \uparrow j-k} M_{1 \rightarrow j} A^{(k) \uparrow j-k}) = k_q \text{tr}_{\hat{F}(2 \dots j)} (\hat{R}_{j-k+1 \leftarrow 1} M_{1 \rightarrow j} A^{(k) \uparrow j-k}) \\ &= k_q \text{tr}_{\hat{F}(2 \dots j)} (A^{(k) \uparrow j-k} \hat{R}_{j-k+1 \leftarrow 1} M_{1 \rightarrow j}) . \end{aligned}$$

We used (12) in the first equality; in the second equality, we moved  $A^{(k)}$  to the left using the cyclic property of the trace. Finally, we obtain

$$\begin{aligned} (M^{\leftarrow j-k} \sigma_k(M))_1 &= (k+1)_q \text{tr}_{\hat{F}(2 \dots j)} (A^{(k+1) \uparrow j-k-1} \hat{R}_{j-k \leftarrow 1} M_{1 \rightarrow j}) \\ &+ k_q \text{tr}_{\hat{F}(2 \dots j)} (A^{(k) \uparrow j-k} \hat{R}_{j-k+1 \leftarrow 1} M_{1 \rightarrow j}) . \end{aligned} \quad (26)$$

Eq.(20) follows from the expressions (26) for  $M^{\leftarrow j-k} \sigma_k(M)$ ,  $k = 1, 2, \dots, j-1$ , and  $M^{\leftarrow j} \sigma_0(M) = M^{\leftarrow j}$ . ■

### 3.4 Corollaries.

**Newton relations.** Taking the trace in (20), we find, for a  $q$ -admissible  $j$ ,

$$q^{-j} j_q \sigma_j(M) = \sum_{k=0}^{j-1} (-1)^{j-k+1} \bar{s}_{j-k}(M) \sigma_k(M) , \quad (27)$$

$$q^j j_q \tau_j(M) = \sum_{k=0}^{j-1} s_{j-k}(M) \tau_k(M) . \quad (28)$$

**Cayley-Hamilton theorem.** Assume that  $\hat{R}$  is even of height  $n$ . Then there exist tensors  $\epsilon_{a_1 a_2 \dots a_n}$  and  $\epsilon^{a_1 a_2 \dots a_n}$  such that

$$(A^{(n)})_{a_1 \dots a_n}^{b_1 \dots b_n} = \epsilon_{a_1 \dots a_n} \epsilon^{b_1 \dots b_n} \quad \text{and} \quad \epsilon_{a_1 \dots a_n} \epsilon^{a_1 \dots a_n} = 1 .$$

By (12) for  $k = n$  and  $i = 0$ , we have

$$(M_{1 \rightarrow n})_{b_1 \dots b_n}^{a_1 \dots a_n} \epsilon^{b_1 \dots b_n} = \det_q(M) \epsilon^{a_1 \dots a_n} , \quad (29)$$

where the *quantum determinant* of the HQ-matrix  $M$  is defined, up to a normalization factor, by

$$\det_q(M) := \epsilon_{a_1 \dots a_n} (M_{1 \rightarrow n})_{b_1 \dots b_n}^{a_1 \dots a_n} \epsilon^{b_1 \dots b_n} .$$

Eq.(29) implies that

$$\tilde{M}^{\wedge n} = \det_q(M) \mathcal{D} ,$$

where the matrix  $\mathcal{D}$  is defined by

$$\mathcal{D}_1 := \text{tr}_{\hat{F}(2 \dots n)}(A^{(n)}) .$$

Taking  $j = n$  in (22) we obtain the Cayley–Hamilton theorem for the half-quantum matrices:

$$\sum_{k=0}^{n-1} (-1)^{n-k} \tilde{M}^{\leftarrow n-k} \sigma_k(M) + n_q \det_q(M) \mathcal{D} = 0 . \quad (30)$$

**Standard  $\hat{R}$ .** Let  $(\hat{R}_{DJ}, P)$  be the compatible pair with the standard multi-parametric Drinfeld–Jimbo R-matrix in dimension  $n$ . In<sup>8</sup>, a version of the Cayley–Hamilton theorem, with usual matrix powers but diagonal matrices as coefficients, has been proven. Namely, the HQ-matrix  $M$  satisfies the matrix identity

$$M^n + \sum_{k=1}^n (-1)^k \Sigma_k M^{n-k} = 0 , \quad (31)$$

with certain diagonal matrices  $\Sigma_m$ ,  $m = 1, \dots, n$ . By the same technics as in<sup>7</sup> one can relate modified powers of  $M$  - in this situation, i.e., for the compatible pair  $(\hat{R}_{DJ}, P)$  - with usual powers and deduce (31) from (30).

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